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Generalized W-class state and its monogamy relation

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Abstract
We generalize the W-class of states from n-qubits to n-qudits and prove that their entanglement is fully characterized by their partial entanglements even for the case of the mixture that consists of a W-class state and a product state $|0\rangle^{\otimes n}$.

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1. Introduction

Quantum entanglement is one of the most non-classical features in quantum mechanics and provides us with a lot of applications. Due to its variety of usages, much attention has been shown, so far, for the quantification of entanglement and, thus, the concept of entanglement measure has been naturally arisen. *Concurrence* [1] is one of the most well-known bipartite entanglement measures with an explicit formula for 2-qubit system while there does not exist any analytic way of evaluation yet for the general case of higher dimensional mixed states. Another entanglement measure that can be considered as a dual to concurrence is the *concurrence of assistance* (CoA) [2], and this can be interpreted as the maximal average concurrence that two parties in the bipartite system can locally prepare with the help of the third party who has the purification of the bipartite system.

In multipartite quantum system, there can be several inequivalent types of entanglement among the subsystems and the amount of entanglement with different types might not be directly comparable to each other. For 3-qubit pure states, it is known that there are two inequivalent classes of genuine tripartite entangled states [3]; one is the Greenberger–Horne–Zeilinger (GHZ) class [4], and the other one is the W-class [3]. This can be characterized by means of stochastic local operations and classical communication (SLOCC), that is, the conversion of the states in a same class can be achieved through local operation and classical communication with non-zero probability.

Another way to characterize the different types of entanglement distribution is by using the *monogamy relation* of entanglement. Unlike classical correlations, the amount of entanglement that can be shared between any of two parties and the others is strongly constrained by the entanglement between the two parties.

In 3-qubit systems, Coffman, Kundu and Wootters (CKW) [5] first introduced a monogamy inequality in terms of a bipartite entanglement measure, concurrence, as

$$\mathcal{C}^2_{A(BC)} \geqslant \mathcal{C}^2_{AB} + \mathcal{C}^2_{AC}, \tag{1}$$

where $\mathcal{C}_{A(BC)} = \mathcal{C}(|\psi\rangle_{A(BC)})$ is the concurrence of a 3-qubit state $|\psi\rangle_{A(BC)}$ for a bipartite cut of subsystems between A and BC and $\mathcal{C}_{AB} = \mathcal{C}(\rho_{AB})$. Its generalization into n-qubit case was also proved [6] and, symmetrically, its dual inequality in terms of the CoA for 3-qubit states,

$$\mathcal{C}^2_{A(BC)} \leqslant \left(\mathcal{C}^a_{AB}\right)^2 + \left(\mathcal{C}^a_{AC}\right)^2, \tag{2}$$

and its generalization into n-qubit cases have been shown in [7, 8].

In 3-qubit systems, two inequivalent classes of genuine tripartite entangled states, GHZ and W-classes, show extreme differences in terms of CKW and its dual inequalities. In other words, CKW and its dual inequalities for 3-qubit pure states are saturated by W-class states, which implies that the differences of the terms in the inequalities are all zero, whereas the differences between terms can assume their largest values for the GHZ-class state.

Here, W-class states are of special interest, since the saturation of the inequality implies that a genuine tripartite entanglement can have a complete characterization by means of the bipartite ones inside it. In other words, the entanglement A-BC, measured by concurrence, is completely determined by its partial entanglements, A-B and A-C. For the case of n-qubit W-class states, generalized CKW and its dual inequalities are also saturated, and thus the same interpretation can be applied.

In this paper, we generalized the concept of W-class states from n-qubits to n-qudits and show that their entanglement is fully characterized by their partial entanglements. We also prove that the complete characterization of the global entanglement in terms of its partial entanglement is possible even for the case of the mixture consisting of a W-class state and a product state $|0\rangle^{\otimes n}$.

This paper is organized as follows. In section 2, we recall the monogamy relation of n-qubit W-class states in terms of CKW and its dual inequalities. In section 3.1, we provide more general monogamy relations of n-qubit W-class states with respect to arbitrary partitions by investigating the structure of n-qubit W-class states. In section 3.2 we generalize the concept of W-class states to arbitrary n-qudit system as well as its monogamy relations with respect to arbitrary partitions. In section 4, we consider the class of multipartite mixed states that is a mixture of a W-class state and a product state. We also provide its monogamy relations in terms of its partial entanglement by studying its structural properties. In section 5 we summarize our results.

2. Monogamy relation of *n*-qubit W-class states

For any bipartite pure state $|\phi\rangle_{AB} \in \mathbb{C}^d \otimes \mathbb{C}^{d'}$, concurrence of $|\phi\rangle_{AB}$ is defined as

$$\mathcal{C}(|\phi\rangle_{AB}) = \sqrt{2\left(1 - \operatorname{tr}\rho_A^2\right)}, \tag{3}$$

where $\rho_A = \operatorname{tr}_B|\phi\rangle_{AB}\langle\phi|$. For any mixed state ρ_{AB}, it is defined as

$$\mathcal{C}(\rho_{AB}) = \min \sum_k p_k \mathcal{C}(|\phi_k\rangle_{AB}), \tag{4}$$

where the minimum is taken over its all possible pure state decompositions, $\rho_{AB} = \sum_k p_k|\phi_k\rangle_{AB}\langle\phi_k|$.

Another entanglement measure that can be considered as a dual to concurrence is CoA [2], which is defined as

$$\mathcal{C}^a(\rho_{AB}) = \max \sum_k p_k \mathcal{C}(|\phi_k\rangle_{AB}), \tag{5}$$

where the maximum is taken over all possible decompositions of ρ_{AB}.

For 3-qubit W-class states

$$|W\rangle_{ABC} = a|100\rangle_{ABC} + b|010\rangle_{ABC} + c|001\rangle_{ABC}, \tag{6}$$

with $|a|^2 + |b|^2 + |c|^2 = 1$, CKW and its dual inequalities (1) and (2) are saturated, that is,

$$\mathcal{C}^2_{A(BC)} = \mathcal{C}^2_{AB} + \mathcal{C}^2_{AC}, \qquad \mathcal{C}_{AB} = \mathcal{C}^a_{AB}, \qquad \mathcal{C}_{AC} = \mathcal{C}^a_{AC}. \tag{7}$$

In other words, the entanglement of W-class states between one party and the rest can have a complete characterization in terms of the *partial entanglement* that is the bipartite entanglement between one party and each of the rest parties.

For n-qubit systems $A_1 \otimes \cdots \otimes A_n$ where $A_i \cong \mathbb{C}^2$ for $i = 1, \ldots, n$, CKW and its dual inequalities can be generalized as [6, 8]

$$\mathcal{C}^2_{A_1(A_2\cdots A_n)} \geqslant \mathcal{C}^2_{A_1A_2} + \cdots + \mathcal{C}^2_{A_1A_n}, \qquad \mathcal{C}^2_{A_1(A_2\cdots A_n)} \leqslant \left(\mathcal{C}^a_{A_1A_2}\right)^2 + \cdots + \left(\mathcal{C}^a_{A_1A_n}\right)^2. \tag{8}$$

For n-qubit W-class states,

$$|W\rangle_{A_1\cdots A_n} = a_1|1\cdots 0\rangle_{A_1\cdots A_n} + \cdots + a_n|0\cdots 1\rangle_{A_1\cdots A_n}, \qquad \sum_{i=1}^n |a_i|^2 = 1, \tag{9}$$

the inequalities (8) are saturated; that is,

$$\mathcal{C}^2_{A_1(A_2\cdots A_n)} = \mathcal{C}^2_{A_1A_2} + \cdots + \mathcal{C}^2_{A_1A_n}, \qquad \mathcal{C}_{A_1A_i} = \mathcal{C}^a_{A_1A_i}, \quad i = 2, \ldots, n. \tag{10}$$

In fact, there can be several ways to show equation (10). Since any 2-qubit reduced density matrix $\rho_{A_1A_i}$ of $|W\rangle_{A_1\cdots A_n}$ can have analytic formulae for concurrence and concurrence of assistance [1, 2], one of the ways to check equalities (10) is using the formulae in [1, 2]. However, there is no formula for the general case of a bipartite quantum state with arbitrary dimension, so here we use the method of considering all possible decompositions of the bipartite mixed state $\rho_{A_1A_i}$ so that the optimization process in (4) and (5) can also be used later in this paper for higher dimensional quantum systems.

Let us first consider $\mathcal{C}^2_{A_1(A_2\cdots A_n)} = 2\left(1 - \operatorname{tr}\rho^2_{A_1}\right)$ where $\rho_{A_1} = \operatorname{tr}_{A_2\cdots A_n}(|W\rangle_{A_1\cdots A_n}\langle W|)$. Since $\rho_{A_1} = |a_1|^2|1\rangle_{A_1}\langle 1| + \left(\sum_{i=2}^n |a_i|^2\right)|0\rangle_{A_1}\langle 0|$, we can easily see

$$\mathcal{C}^2_{A_1(A_2\cdots A_n)} = 4|a_1|^2\left(\sum_{i=2}^n |a_i|^2\right). \tag{11}$$

For $\mathcal{C}^2_{A_1A_i}$ with $i \in \{2, \ldots, n\}$, let us consider

$$\begin{aligned}\rho_{A_1A_i} &= \operatorname{tr}_{A_2\cdots\widehat{A}_i\cdots A_n}(|W\rangle_{A_1\cdots A_n}\langle W|) = (a_1|10\rangle + a_i|01\rangle)_{A_1A_i}(a_1^*\langle 10| + a_i^*\langle 01|)\\ &\quad + (|a_2|^2 + \cdots + |\widehat{a}_i|^2 + \cdots + |a_n|^2)|00\rangle_{A_1A_i}\langle 00|,\end{aligned} \tag{12}$$

where $A_2\cdots\widehat{A}_i\cdots A_n = A_2\cdots A_{i-1}A_{i+1}\cdots A_n$ and $(|a_2|^2 + \cdots + |\widehat{a}_i|^2 + \cdots + |a_n|^2) = (|a_2|^2 + \cdots + |a_{i-1}|^2 + |a_{i+1}|^2 + \cdots + |a_n|^2)$. Now, let $|\tilde{x}\rangle_{A_1A_i} = a_1|10\rangle_{A_1A_i} + a_i|01\rangle_{A_1A_i}$ and $|\tilde{y}\rangle_{A_1A_i} = \sqrt{|a_2|^2 + \cdots + |\widehat{a}_i|^2 + \cdots |a_n|^2}|00\rangle_{A_1A_i}$, where $|\tilde{x}\rangle_{A_1A_i}$ and $|\tilde{y}\rangle_{A_1A_i}$ are unnormalized states of the subsystems A_1A_i. Then, by the Hughston–Jozsa–Wootters (HJW) theorem [9], any other decomposition of $\rho_{A_1A_i} = \sum_{h=1}^r |\tilde{\phi}_h\rangle_{A_1A_i}\langle\tilde{\phi}_h|$ with size $r \geqslant 2$ can be obtained by an $r \times r$ unitary matrix (u_{hl}) where

$$|\tilde{\phi}_h\rangle_{A_1A_i} = u_{h1}|\tilde{x}\rangle_{A_1A_i} + u_{h2}|\tilde{y}\rangle_{A_1A_i}. \tag{13}$$

Let $\rho_{A_1A_i} = \sum_h p_h|\phi_h\rangle_{A_1A_i}\langle\phi_h|$ where $\sqrt{p_h}|\phi_h\rangle_{A_1A_i} = |\tilde{\phi}_h\rangle_{A_1A_i}$ and $p_h = |\langle\tilde{\phi}_h|\tilde{\phi}_h\rangle|$; then, by straightforward calculation, we can easily see that the average concurrence of the pure state decomposition $\rho_{A_1A_i} = \sum_h p_h|\phi_h\rangle_{A_1A_i}\langle\phi_h|$ is

$$\sum_{h=1}^{r} p_h\mathcal{C}(|\phi_h\rangle_{A_1A_i}) = \sum_{h=1}^{r} p_h\sqrt{2\big(1-\mathrm{tr}\big(\rho_{A_1}^h\big)^2\big)} = 2|a_1||a_i|, \tag{14}$$

where $\rho_{A_1}^h = \mathrm{tr}_{A_i}(|\phi_h\rangle_{A_1A_i}\langle\phi_h|)$. In other words, the average concurrence remains the same for any pure state decomposition of $\rho_{A_1A_i}$, and thus, we can have

$$\min\sum_h p_h\mathcal{C}(|\phi_h\rangle_{A_1A_i}) = \max\sum_h p_h\mathcal{C}(|\phi_h\rangle_{A_1A_i}) \tag{15}$$

where the maximum and minimum are taken over all possible decomposition of $\rho_{A_1A_2}$. This implies

$$\mathcal{C}_{A_1A_i} = \mathcal{C}^a_{A_1A_i} = 2|a_1||a_i|, \tag{16}$$

which leads to (10).

Whereas procedures to realize pure state ensembles are well known, it has not been typical practice in actual evaluations of optimization problems such as minimization and maximization in concurrence and concurrence of assistance, respectively. The obstacle to taking this theoretical knowledge to practical use has been due to the huge amount of calculation with respect to Hilbert space dimension. However, our result shows that this increasing amount of calculation can sometimes be simplified so that analytic solutions for concurrence and concurrence of assistance can indeed be obtained in a reasonable amount of computational time, even though no general solution exists except for 2-qubit states. We believe that our new calculation technique can also be applied in many other related areas, especially where numerical tests are required.

3. General monogamy relation of multipartite W-class states

In section 2, we have seen the monogamy relations of n-qubit W-class states between subsystems in terms of CKW and its dual inequalities. Here, we investigate the structure of W-class states in n-qubit system by considering arbitrary partitions of subsystems and derive more general concept of monogamy relations between the parties. Furthermore, we generalize the concept of W-class states to arbitrary n-qudit system and also consider the monogamy relations in terms of arbitrary partitions.

3.1. Structure of n-qubit W-class states

For n-qubit W-class states,

$$|W\rangle_{A_1\cdots A_n} = a_0|0\cdots 1\rangle_{A_1\cdots A_n} + \cdots + a_{n-1}|1\cdots 0\rangle_{A_1\cdots A_n}, \qquad \sum_{i=0}^{n-1}|a_i|^2 = 1, \tag{17}$$

let us consider a partition $P = \{P_1,\ldots,P_m\}, m \leqslant n$ for the set of subsystems $S = \{A_1,\ldots,A_n\}$ where each of P_s is a subset of S, such that,

$$|P_s| = n_s, \qquad \sum_s n_s = n, \qquad P_s\cap P_t = \emptyset \quad \text{for } s\neq t, \qquad \bigcup_s P_s = S. \tag{18}$$

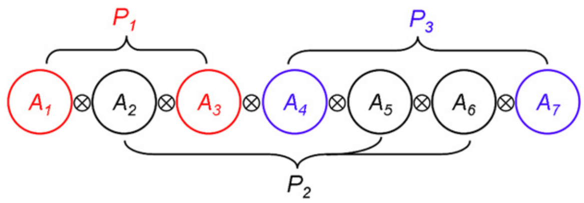

Figure 1. A partition for the set of subsystem $S = \{A_1, \ldots A_7\}$ where $P_1 = \{A_1, A_3\}$, $P_2 = \{A_2, A_5, A_6\}$ and $P_3 = \{A_4, A_7\}$.

(This figure is in colour only in the electronic version)

For simplicity, let us first consider the case when $P = \{P_1, P_2, P_3\}$ with $|P_s| = n_s$ where $s \in \{1, 2, 3\}$. Figure 1 shows an example of partition for a set of subsystems $S = \{A_1, \ldots A_7\}$ where $P_1 = \{A_1, A_3\}$, $P_2 = \{A_2, A_5, A_6\}$ and $P_3 = \{A_4, A_7\}$.

Without loss of generality, we may assume $P_1 = \{A_1, \ldots, A_{n_1}\}$, $P_2 = \{A_{n_1+1}, \ldots, A_{n_1+n_2}\}$ and $P_2 = \{A_{n_1+n_2+1}, \ldots, A_n\}$; otherwise we can have some proper reordering of the subsystems.

Here, we use the representation, that is,

$$|0\cdots 1\cdots 0\rangle_{P_s} = |2^i\rangle_{P_s} \tag{19}$$

where $|0\cdots 1\cdots 0\rangle_{P_s}$ is an n_s-qubit product state of the party P_s whose ith subsystem from the right is 1 and 0 elsewhere. Then (9) can be rewritten as

$$|W\rangle_{P_1P_2P_3} = |\tilde{x}\rangle_{P_1}|\vec{0}\rangle_{P_2}|\vec{0}\rangle_{P_3} + |\vec{0}\rangle_{P_1}|\tilde{y}\rangle_{P_2}|\vec{0}\rangle_{P_3} + |\vec{0}\rangle_{P_1}|\vec{0}\rangle_{P_2}|\tilde{z}\rangle_{P_3}, \tag{20}$$

where $|\vec{0}\rangle_{P_s} = |0\cdots 0\rangle_{P_s}$ and $|\tilde{x}\rangle_{P_1}$, $|\tilde{y}\rangle_{P_2}$ and $|\tilde{z}\rangle_{P_3}$ are unnormalized states in P_1, P_2 and P_3 respectively such that $|\tilde{x}\rangle_{P_1} = \sum_{j=0}^{n_1-1} a_{n_3+n_2+j}|2^j\rangle_{P_1}$, $|\tilde{y}\rangle_{P_2} = \sum_{k=0}^{n_2-1} a_{n_3+k}|2^k\rangle_{P_2}$ and $|\tilde{z}\rangle_{P_3} = \sum_{l=0}^{n_3-1} a_l|2^l\rangle_{P_3}$.

Here, we note that $|\tilde{x}\rangle_{P_1}$, $|\tilde{y}\rangle_{P_2}$ and $|\tilde{z}\rangle_{P_3}$ are unnormalized W-class states of the parties P_1, P_2 and P_3 respectively. Thus any n-qubit W-class state can have this type of representation, that is,

$$|W\rangle_{P_1P_2P_3} = \sqrt{q_1}|W\rangle_{P_1}|\vec{0}\rangle_{P_2}|\vec{0}\rangle_{P_3} + \sqrt{q_2}|\vec{0}\rangle_{P_1}|W\rangle_{P_2}|\vec{0}\rangle_{P_3} + \sqrt{q_3}|\vec{0}\rangle_{P_1}|\vec{0}\rangle_{P_2}|W\rangle_{P_3}, \tag{21}$$

where $q_1 = \sum_{j=0}^{n_1-1} |a_{n_3+n_2+j}|^2$, $q_2 = \sum_{k=0}^{n_2-1} |a_{n_3+k}|^2$ and $q_3 = \sum_{l=0}^{n_3-1} |a_l|^2$ with the normalization condition $q_1 + q_2 + q_3 = 1$.

If we just rename $|W\rangle_{P_s} = |1\rangle_{P_s}$ and $|\vec{0}\rangle_{P_s} = |0\rangle_{P_s}$, then $|1\rangle_{P_s}$ and $|0\rangle_{P_s}$ are orthogonal to each other, and (21) can be rewritten as

$$|W\rangle_{P_1P_2P_3} = \sqrt{q_1}|1\rangle_{P_1}|0\rangle_{P_2}|0\rangle_{P_3} + \sqrt{q_2}|0\rangle_{P_1}|1\rangle_{P_2}|0\rangle_{P_3} + \sqrt{q_3}|0\rangle_{P_1}|0\rangle_{P_2}|1\rangle_{P_3}, \tag{22}$$

which is a tripartite W-class state in $(\mathbb{C}^2)^{n_1} \otimes (\mathbb{C}^2)^{n_2} \otimes (\mathbb{C}^2)^{n_3}$ quantum systems.

Similarly, for an arbitrary partition $P = \{P_1, \ldots, P_m\}$ of size m, we can have

$$\begin{aligned} |W\rangle_{P_1\cdots P_m} &= \sqrt{q_1}|W\rangle_{P_1}|0\rangle_{P_2}\cdots|0\rangle_{P_m} + \sqrt{q_2}|0\rangle_{P_1}|W\rangle_{P_2}\cdots|0\rangle_{P_3} \\ &\quad + \cdots + \sqrt{q_m}|0\rangle_{P_1}|0\rangle_{P_2}\cdots|W\rangle_{P_m} \\ &= \sqrt{q_1}|1\rangle_{P_1}|0\rangle_{P_2}\cdots|0\rangle_{P_m} + \sqrt{q_2}|0\rangle_{P_1}|1\rangle_{P_2}\cdots|0\rangle_{P_3} \\ &\quad + \cdots + \sqrt{q_m}|0\rangle_{P_1}|0\rangle_{P_2}\cdots|1\rangle_{P_m}. \end{aligned} \tag{23}$$

For any partition $P = \{P_1, \ldots, P_m\}$ of the set of subsystems $S = \{A_1, \ldots, A_n\}$, the n-qubit W-class state (17) can be also considered as an m-partite W-class state with different names of the basis, and thus we can have following lemma.

Lemma 1. *For any n-qubit W-class states* $|W\rangle_{A_1\cdots A_n}$ *and a partition* $P=\{P_1,\ldots,P_m\}$ *of the set of subsystems* $S=\{A_1,\ldots,A_n\}$,

$$\mathcal{C}^2_{P_s(P_1\cdots\widehat{P}_s\cdots P_m)}=\sum_{k\neq s}\mathcal{C}^2_{P_sP_k}=\sum_{k\neq s}\left(\mathcal{C}^a_{P_sP_k}\right)^2, \tag{24}$$

and

$$\mathcal{C}_{P_sP_k}=\left(\mathcal{C}^a_{P_sP_k}\right), \tag{25}$$

for all $k\neq s$.

3.2. *n-qudit W-class states*

Now, we generalize the concept of W-class states to arbitrary n-qudit systems with similar properties of monogamy relations as in lemma 1.

First, we would like to mention that the problem of qubit-to-qudit extensions, the qudit case for $d>2$, contains many distinguishable and nontrivial features compared to the qubit case. For example the CKW inequality for qubit monogamy of entanglement is not generally true for higher dimensional quantum system, e.g. the recent counter example for the violation of CKW inequality in a three-qutrit system [10]. Furthermore, we can also find another counter example in $\mathbb{C}^3\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ quantum systems[1], thereby showing that the CKW inequality is strictly true only for qubit-based quantum systems: even a tiny extension in any of the subsystems leads to a violation.

Even though we consider concurrence and concurrence of assistance as entanglement measures similar to the qubit case, our result shows a new relation of entanglement for higher dimensional quantum systems. Our result shows a fundamental step of the challenges to the richness of entanglement studies for system of higher dimensions and multipartite systems.

Now, let us recall the structure of n-qubit W-class states in (17). We can easily observe that equation (17) is a coherent superposition of all possible product states that are 'a single local unitary operation away' from all zero states $|0\rangle^{\otimes n}$. In other words, the first term $|0\cdots1\rangle$ can be obtained from $|0\rangle^{\otimes n}$ by applying the 1-qubit Pauli operator σ_x on the last qubit, that is,

$$|0\cdots1\rangle=I\otimes I\otimes\cdots\otimes\sigma_x|0\cdots0\rangle, \tag{26}$$

and the rest can be obtained in a similar way.

Let us consider a class of n-qudit quantum states,

$$\left|W_n^d\right\rangle_{A_1\cdots A_n}=\sum_{i=1}^{d-1}(a_{1i}|i\cdots0\rangle_{A_1\cdots A_n}+\cdots+a_{ni}|0\cdots i\rangle_{A_1\cdots A_n}), \tag{27}$$

with $\sum_{s=1}^{n}\sum_{i=1}^{d-1}|a_{si}|^2=1$. Equation (27) is a coherent superposition of all possible product states that are 'a single local transposition away' from all zero states $|0\rangle^{\otimes n}$, and in the case $d=2$, it is reduced to n-qubit W-class states in (17).

Now, we will see that the class of states (27) has similar properties as in equation (10); that is,

$$\mathcal{C}^2_{A_1(A_2\cdots A_n)}=\mathcal{C}^2_{A_1A_2}+\cdots+\mathcal{C}^2_{A_1A_n},\qquad \mathcal{C}_{A_1A_t}=\mathcal{C}^a_{A_1A_t}, t=2,\ldots,n. \tag{28}$$

[1] Consider $|\psi\rangle_{ABC}=\frac{1}{\sqrt{6}}(\sqrt{2}|010\rangle+\sqrt{2}|101\rangle+|200\rangle+|211\rangle)$ in $\mathbb{C}^3\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ quantum systems. It can be easily seen that $\mathcal{C}^2_{A(BC)}=\frac{12}{9}$ while $\mathcal{C}^2_{AB}=\mathcal{C}^2_{AC}=\frac{8}{9}$ which implies the violation of CKW inequality.

In other words, CKW and its dual inequalities are saturated by the class of states (27). To see this, let us first consider $\mathcal{C}^2_{A_1(A_2\cdots A_n)} = 2(1-\mathrm{tr}\rho^2_{A_1})$ where $\rho_{A_1} = \mathrm{tr}_{A_2\cdots A_n}\left(\left|W^d_n\right\rangle_{A_1\cdots A_n}\left\langle W^d_n\right|\right)$. Since $\rho_{A_1} = |\tilde{\psi}_1\rangle_{A_1}\langle\tilde{\psi}_1| + \sum_{s=2}^n \alpha_s^2|0\rangle_{A_1}\langle 0|$,

$$\mathcal{C}^2_{A_1(A_2\cdots A_n)} = 4\alpha_1^2\left(\sum_{s=2}^n \alpha_s^2\right), \tag{29}$$

where $|\tilde{\psi}_s\rangle_{A_s} = \sum_{i=1}^{d-1} a_{si}|i\rangle_{A_s}$ is an unnormalized state of subsystem A_s for $s \in \{1,\ldots,n\}$ and $|\langle\tilde{\psi}_s|\tilde{\psi}_s\rangle| = \sum_{i=1}^{d-1}|a_{si}|^2 = \alpha_s^2$, with the normalization condition $\sum_{s=1}^n \alpha_s^2 = 1$.

For $\mathcal{C}^2_{A_1A_t}$ with $t \in \{2,\ldots,n\}$, let us consider

$$\begin{aligned}\rho_{A_1A_t} &= \mathrm{tr}_{A_2\cdots\widehat{A}_t\cdots A_n}\left(\left|W^d_n\right\rangle_{A_1\cdots A_n}\left\langle W^d_n\right|\right)\\ &= \sum_{i,j=1}^{d-1}(a_{1i}|i0\rangle + a_{ti}|0i\rangle)_{A_1A_t}(a^*_{1j}\langle j0| + a^*_{tj}\langle 0j|)\\ &\quad + \left(\alpha_2^2+\cdots+\widehat{\alpha}_t^2+\cdots+\alpha_n^2\right)|00\rangle_{A_1A_t}\langle 00|,\end{aligned} \tag{30}$$

where $A_2\cdots\widehat{A}_i\cdots A_n = A_2\cdots A_{i-1}A_{i+1}\cdots A_n$ and $\left(\alpha_2^2+\cdots+\widehat{\alpha}_t^2+\cdots+\alpha_n^2\right) = \left(\alpha_2^2+\cdots+\alpha_{t-1}^2+\alpha_{t+1}^2+\cdots+\alpha_n^2\right)$.

Now, let us denote $|\tilde{x}\rangle_{A_1A_t} = \sum_{i=1}^{d-1}(a_{1i}|i0\rangle + a_{ti}|0i\rangle)_{A_1A_t}$ and $|\tilde{y}\rangle_{A_1A_t} = \sqrt{\alpha_2^2+\cdots+\widehat{\alpha}_t^2+\cdots+\alpha_n^2}|00\rangle_{A_1A_t}$, where $|\tilde{x}\rangle_{A_1A_t}$ and $|\tilde{y}\rangle_{A_1A_t}$ are unnormalized states of the subsystems A_1A_t. Then, by the HJW theorem, any other decomposition of $\rho_{A_1A_t} = \sum_{h=1}^r|\tilde{\phi}_h\rangle_{A_1A_t}\langle\tilde{\phi}_h|$ with size $r \geqslant 2$ can be obtained by an $r\times r$ unitary matrix (u_{hl}) where

$$|\tilde{\phi}_h\rangle_{A_1A_t} = u_{h1}|\tilde{x}\rangle_{A_1A_t} + u_{h2}|\tilde{y}\rangle_{A_1A_t}. \tag{31}$$

Let $\rho_{A_1A_t} = \sum_h p_h|\phi_h\rangle_{A_1A_t}\langle\phi_h|$ where $\sqrt{p_h}|\phi_h\rangle_{A_1A_t} = |\tilde{\phi}_h\rangle_{A_1A_t}$ and $p_h = |\langle\tilde{\phi}_h|\tilde{\phi}_h\rangle|$; then, after a tedious calculation, we can see that the average concurrence of the pure state decomposition $\rho_{A_1A_t} = \sum_h p_h|\phi_h\rangle_{A_1A_t}\langle\phi_h|$ is

$$\begin{aligned}\sum_{h=1}^r p_h\mathcal{C}(|\phi_h\rangle_{A_1A_t}) &= \sum_{h=1}^r p_h\sqrt{2\left(1-\mathrm{tr}\left(\rho^h_{A_1}\right)^2\right)}\\ &= 2\alpha_1\alpha_t,\end{aligned} \tag{32}$$

where $\rho^h_{A_1} = \mathrm{tr}_{A_t}(|\phi_h\rangle_{A_1A_t}\langle\phi_h|)$. Similar to the n-qubit case, the average concurrence remains the same for any pure state decomposition of $\rho_{A_1A_t}$, and thus, we can have

$$\min\sum_h p_h\mathcal{C}(|\phi_h\rangle_{A_1A_t}) = \max\sum_h p_h\mathcal{C}(|\phi_h\rangle_{A_1A_t}), \tag{33}$$

where the maximum and minimum are taken over all possible decomposition of $\rho_{A_1A_t}$. This implies

$$\mathcal{C}_{A_1A_t} = \mathcal{C}^a_{A_1A_t} = 2\alpha_1\alpha_t, \tag{34}$$

which leads to (28).

Now, let us consider a partition $P = \{P_1,\ldots,P_m\}, m \leqslant n$ where each of P_s with $s \in \{1,\ldots,m\}$ contains several qudits such that $|P_s| = n_s$ and $n_1+\cdots+n_m = n$. Without loss of generality, we may assume $P_1 = \{A_1,\ldots,A_{n_1}\}$, $P_2 = \{A_{n_1+1},\ldots,A_{n_1+n_2}\},\ldots,P_m = \{A_{n_1+\cdots+n_{m-1}+1},\ldots,A_n\}$; otherwise we can have some proper reordering of the subsystems. For each party P_s of the partition P, let

$$|\tilde{x}_{si}\rangle_{P_s} = a_{(n_1+\cdots+n_{s-1}+1)i}|i0\cdots 0\rangle_{P_s} + a_{(n_1+\cdots+n_{s-1}+2)i}|0i\cdots 0\rangle_{P_s} + \cdots + a_{(n_1+\cdots+n_s)i}|00\cdots i\rangle_{P_s}, \tag{35}$$

then $|\tilde{x}_{si}\rangle_{P_s}$ is an unnormalized state of the party P_s and (27) can be rewritten as

$$\left|W_n^d\right\rangle_{P_1\cdots P_m} = \sum_{i=1}^{d-1}(|\tilde{x}_{1i}\rangle_{P_1} \otimes |\vec{0}\rangle_{P_2} \otimes \cdots \otimes |\vec{0}\rangle_{P_m} + \cdots + |\vec{0}\rangle_{P_1} \otimes |\vec{0}\rangle_{P_2} \otimes \cdots \otimes |\tilde{x}_{mi}\rangle_{P_m}), \quad (36)$$

where $|\vec{0}\rangle_{P_s} = |0\cdots0\rangle_{P_s}$. If we consider the normalized state $|x_{si}\rangle_{P_s} = \frac{1}{\sqrt{q_{si}}}|\tilde{x}_{si}\rangle_{P_s}$ with $|\langle\tilde{x}_{si}|\tilde{x}_{si}\rangle| = q_{si}^2$ and rename $|x_{si}\rangle_{P_s} = |i\rangle_{P_s}$ and $|\vec{0}\rangle_{P_s} = |0\rangle_{P_s}$, then (36) can be represented as

$$\left|W_n^d\right\rangle_{P_1\cdots P_m} = \sum_{i=1}^{d-1}(\sqrt{q_{1i}}|i\rangle_{P_1} \otimes |0\rangle_{P_2} \otimes \cdots \otimes |0\rangle_{P_m} + \cdots + \sqrt{q_{mi}}|0\rangle_{P_1} \otimes |0\rangle_{P_2} \otimes \cdots \otimes |i\rangle_{P_m}), \quad (37)$$

which is an m-partite generalized W-class state and, thus, we can have the second lemma which incorporates lemma 1.

Lemma 2. *For any n-qudit generalized W-class states $|W\rangle_{A_1\cdots A_n}$ in (27) and a partition $P = \{P_1, \ldots, P_m\}$ for the set of subsystems $S = \{A_1, \ldots, A_n\}$,*

$$\mathcal{C}^2_{P_s(P_1\cdots\hat{P}_s\cdots P_m)} = \sum_{k\neq s}\mathcal{C}^2_{P_sP_k} = \sum_{k\neq s}\left(\mathcal{C}^a_{P_sP_k}\right)^2, \quad (38)$$

and

$$\mathcal{C}_{P_sP_k} = \left(\mathcal{C}^a_{P_sP_k}\right), \quad (39)$$

for all $k \neq s$.

Furthermore, if we consider the state $|\tilde{x}_s\rangle_{P_s}$ of the partition P_s such that

$$|\tilde{x}_s\rangle_{P_s} = \sum_{i=1}^{d-1}(a_{(n_1+\cdots+n_{s-1}+1)i}|i0\cdots0\rangle_{P_s} + a_{(n_1+\cdots+n_{s-1}+2)i}|0i\cdots0\rangle_{P_s} + \cdots + a_{(n_1+\cdots+n_s)i}|00\cdots i\rangle_{P_s}), \quad (40)$$

then $|\tilde{x}_s\rangle_{P_s}$ is an unnormalized W-class state of n_s-qudit system. Let $|\tilde{x}_s\rangle_{P_s} = \sqrt{q_s}\left|W^d_{n_s}\right\rangle_{P_s}$ with $q_s^2 = \sum_{i=1}^{d-1} q_{si}^2$; then (36) can be also represented as

$$|W\rangle_{P_1\cdots P_m} = \sqrt{q_1}\left|W^d_{n_1}\right\rangle_{P_1} \otimes |0\rangle_{P_2} \otimes \cdots \otimes |0\rangle_{P_m} + \cdots + \sqrt{q_m}|0\rangle_{P_1} \otimes |0\rangle_{P_2} \otimes \cdots \otimes \left|W^d_{n_m}\right\rangle_{P_m}, \quad (41)$$

which is the same type of representation as the n-qubit W-class states in (23).

4. *n*-qudit mixed states

Concurrence is one of most well-known entanglement measures for bipartite quantum system with an explicit formula for 2-qubit system. However, in higher dimensional quantum system, there does not exist any explicit way of evaluation yet for mixed state. The lack of an analytic evaluation technique is mostly due to the difficulty of optimization problem which is minimizing over all possible pure state decompositions of the given mixed state. The difficulty of optimization problem for its dual, CoA, also arises in forms of maximization.

Recently, the optimal pure state decomposition for the mixture of generalized GHZ and W states in 3-qubit system was found [11, 12], and the optimal decomposition, here, was assured by the saturation of CKW inequality. In [13], another monogamy relation in terms of the higher-tangle, the squared concurrence of pure states and its convex-roof extension for mixed states, was also investigated for a mixture of n-qubit W-class state and a product state $|0\rangle^{\otimes n}$.

Here, we investigate the structure of the mixed states that consists of n-qubit W-class states and the product state $|0\rangle^{\otimes n}$ and provide an analytic proof for its saturation of CKW and its dual in equality. This saturation of the inequalities is also true for any partition of the set of subsystems. Noting that the average of squared concurrences is always an upper bound of the square of average concurrences, the result in [13] becomes a special case of the result here.

For any n-qudit W-class state in (27), if we consider the reduced density matrix of the subsystem $\{A_{s_1}, \ldots, A_{s_l}\}$ for $2 \leqslant l \leqslant n-1$, we can easily check that it is always a mixture of some l-qudit W-class state and a product state $|0\cdots 0\rangle$, that is,

$$\rho_{A_{s_1}\cdots A_{s_l}} = p\left|W_l^d\right\rangle_{A_{s_1}\cdots A_{s_l}}\left\langle W_l^d\right| + (1-p)|0\cdots 0\rangle_{A_{s_1}\cdots A_{s_l}}\langle 0\cdots 0|, \tag{42}$$

for some $0 \leqslant p \leqslant 1$.

Conversely, let us consider any mixture of an n-qudit W-class state in (27) and a product state $|0\cdots 0\rangle_{A_1\cdots A_n}$,

$$\rho_{A_1\cdots A_n} = p\left|W_n^d\right\rangle_{A_1\cdots A_n}\left\langle W_n^d\right| + (1-p)|0\cdots 0\rangle_{A_1\ldots A_n}\langle 0\cdots 0|. \tag{43}$$

Since $\rho_{A_1\cdots A_n}$ is an operator of rank 2, we can always have a purification $|\psi\rangle_{A_1\cdots A_n A_{n+1}} \in (\mathbb{C}^d)^{\otimes n+1}$ of $\rho_{A_1\cdots A_n}$ such that

$$|\psi\rangle_{A_1\cdots A_n A_{n+1}} = \sqrt{p}\left|W_n^d\right\rangle_{A_1\cdots A_n} \otimes |0\rangle_{A_{n+1}} + \sqrt{1-p}|0\cdots 0\rangle_{A_1\cdots A_n} \otimes |x\rangle_{A_{n+1}}, \tag{44}$$

where $|x\rangle_{A_{n+1}} = \sum_{1=i}^{d-1} a_{n+1i}|i\rangle_{A_{n+1}}$ is a 1-qudit quantum state of A_{n+1} which is orthogonal to $|0\rangle_{A_{n+1}}$ where $\sum_{1=i}^{d-1} |a_{n+1i}|^2 = 1$. Now, we can easily see that (44) can be rewritten as

$$\begin{aligned}|\psi\rangle_{A_1\cdots A_{n+1}} = \sum_{i=1}^{d-1}[&\sqrt{p}(a_{1i}|i\cdots 00\rangle_{A_1\cdots A_{n+1}} + \cdots + a_{ni}|0\cdots i0\rangle_{A_1\cdots A_{n+1}})\\ &+\sqrt{1-p}a_{n+1i}|0\cdots 0i\rangle_{A_1\cdots A_{n+1}}],\end{aligned} \tag{45}$$

and this is an $(n+1)$-qudit W-class state.

In other words, the reduced density matrix of a generalized W-class state onto any subsystem is a mixture of a W-class state and a product state. Furthermore, any mixture of a W-class state and a product state $|0\cdots 0\rangle$ can be considered as a reduced density matrix of some W-class state in a quantum system with a larger number of parties.

Then, we can have following theorem.

Theorem 1. *Let $\rho_{A_1\cdots A_n}$ be an n-qudit mixed state in $\mathcal{B}((\mathbb{C}^d)^{\otimes n})$, which is a mixture of a generalized W-class state $|W\rangle_{A_1\cdots A_n}$ and a product state $|0\cdots 0\rangle_{A_1\cdots A_n}$ with any weighting factor $0 \leqslant p \leqslant 1$ such that*

$$\rho_{A_1\cdots A_n} = p\left|W_n^d\right\rangle_{A_1\cdots A_n}\left\langle W_n^d\right| + (1-p)|0\cdots 0\rangle_{A_1\cdots A_n}\langle 0\cdots 0|. \tag{46}$$

Then

$$\mathcal{C}^2_{A_1(A_2\cdots A_n)} = \mathcal{C}^2_{A_1A_2} + \cdots + \mathcal{C}^2_{A_1A_n}, \qquad \mathcal{C}_{A_1A_t} = \mathcal{C}^a_{A_1A_t}, \quad t = 2, \ldots, n. \tag{47}$$

Furthermore, for any partition $P = \{P_1, \ldots, P_m\}$ of the set of subsystems $S = \{A_1, \ldots, A_n\}$,

$$\mathcal{C}^2_{P_s(P_1\cdots\hat{P}_s\cdots P_m)} = \sum_{k\neq s}\mathcal{C}^2_{P_sP_k} = \sum_{k\neq s}\left(\mathcal{C}^a_{P_sP_k}\right)^2, \tag{48}$$

and

$$\mathcal{C}_{P_sP_k} = \left(\mathcal{C}^a_{P_sP_k}\right), \tag{49}$$

for all $k \neq s$.

Proof. Let (45) be a purification of $\rho_{A_1\cdots A_n}$ in $(\mathbb{C}^d)^{\otimes n+1}$, then, by lemma 2, we can have

$$\begin{aligned}\mathcal{C}^2_{A_1[(A_2\cdots A_n)A_{n+1}]} &= \mathcal{C}^2_{A_1(A_2\cdots A_n)} + \mathcal{C}^2_{A_1A_{n+1}}\\ &= \sum_{s=2}^{n}\mathcal{C}^2_{A_1A_s} + \mathcal{C}^2_{A_1A_{n+1}},\end{aligned} \tag{50}$$

and, thus,

$$\mathcal{C}^2(\rho_{A_1(A_2\cdots A_n)}) = \mathcal{C}^2_{A_1(A_2\cdots A_n)} = \sum_{s=2}^{n}\mathcal{C}^2_{A_1A_s}. \tag{51}$$

Furthermore, for any partition $P = \{P_1, \ldots, P_m\}$ of the set of subsystems $S = \{A_1, \ldots, A_n\}$,

$$\begin{aligned}\mathcal{C}^2_{P_i[(P_2\cdots\hat{P}_i\cdots P_m)A_{n+1}]} &= \mathcal{C}^2_{P_i(P_2\cdots\hat{P}_i\cdots P_m)} + \mathcal{C}^2_{P_iA_{n+1}}\\ &= \sum_{s\neq i}\mathcal{C}^2_{P_iP_s} + \mathcal{C}^2_{P_iA_{n+1}},\end{aligned} \tag{52}$$

and we can have

$$\mathcal{C}^2_{P_i(P_2\cdots\hat{P}_i\cdots P_m)} = \sum_{s\neq i}\mathcal{C}^2_{P_iP_s}. \tag{53}$$

□

Note that theorem 1 encapsulates the first two lemmas. In other words, if $p = 1$, theorem 1 deals with the monogamy relations of n-qudit W-class states that were presented in lemma 2, and for the case $p = 1, d = 2$, it is about the monogamy relations of n-qubit W-class states that were presented in lemma 1. Thus, theorem 1 deals with the most general case of high-dimensional multipartite mixed states, so far, whose entanglement is completely characterized by their partial entanglements.

5. Conclusions

We have investigated general monogamy relations for W-class states by considering the structure of W-class states in terms of arbitrary partitions of subsystems. We have generalized the concept of W-class states from n-qubit systems to arbitrary n-qudit systems and have shown that their entanglement is completely characterized by their partial entanglements in terms of any partition of the set of subsystems. The structural properties of a class of mixed states that consist of an n-qubit W-class state and a product state $|0\rangle^{\otimes n}$ have been shown, and we have proved that the monogamy relations of the mixture have the same characterization as the case of W-class states.

The structural properties of W-class states by considering an arbitrary partition of the set of subsystems show that the structure of W-class states is inherent with respect to any arbitrary partition by the choice of a proper basis. Our novel technique can also be helpful to study the structural properties of other genuine multipartite entangled states, such as n-qudit GHZ-class states and cluster states [14]. Noting the importance of the study on high-dimensional multipartite entanglement, although there have not been much so far, our results can provide a rich reference for future work on the characterization of multipartite entanglement.

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